

PROBLEMA LINEARE DEI MINIMI QUADRATI

$$Ax = b$$

$$A \in \mathbb{C}^{m \times n} \quad m > n$$

$$b \in I(A) \Rightarrow \exists x^* : Ax^* = b$$

si cerca

$$Ax^* \simeq b$$

cioè si considera

$$d(x) = Ax - b \quad \text{vettore residuo}$$

si vuole minimizzare $d(x)$

si usa

$$\min_{x \in \mathbb{C}^n} \|d(x)\|_2^2 = \|d(x^*)\|_2^2 \leq \|d(x)\|_2^2, \quad \forall x \in \mathbb{C}^n$$

PROPRIETÀ

$$A \in \mathbb{C}^{m \times n} \quad (m > n)$$

$$A^H \in \mathbb{C}^{n \times m}$$

$$I(A) = \{y \in \mathbb{C}^m \mid y = Ax, x \in \mathbb{C}^n\} \subset \mathbb{C}^m$$

$$N(A) = \{z \in \mathbb{C}^n \mid Az = 0\} \subset \mathbb{C}^n$$

$$N(A^H) = \{z \in \mathbb{C}^m \mid A^H z = 0\} \subset \mathbb{C}^m$$

$$I(A)^\perp = N(A^H)$$

dimostr.

$$1) z \in I(A)^\perp$$

$$z^H Ax = 0 \rightarrow (z^H Ax)^H = 0 \rightarrow x^H A^H z = 0$$

$$\rightarrow A^H z = 0 \rightarrow z \in N(A^H)$$

$$2) z \in N(A^H)$$

$$A^H z = 0 \rightarrow x^H A^H z = 0 \quad (x^H A^H z)^H = 0$$

$$z^H Ax = 0 \rightarrow z \in I(A)^\perp$$

$$d(x) = Ax - b$$

$$d(x) = Ax - (b' + b'') \quad \text{con } b' \in I(A), \quad b'' \in I(A)^\perp$$

$$I(A) \cap I(A)^\perp = \{0\} \Rightarrow I(A) \oplus I(A)^\perp = \mathbb{C}^m$$

$$\forall b \in \mathbb{C}^m \quad b = b' + b'' \quad \text{si può scrivere in un modo solo}$$

$$\text{pongo: } y = Ax - b'$$

$$y \in I(A)$$

$$b'' \in I(A)^\perp$$

$$d(x) = y - b''$$

$$\begin{aligned} \|d(x)\|_2^2 &= d(x)^H d(x) = (y - b'')^H (y - b'') = (y^H - b''^H)(y - b'') = \\ &= y^H y - \underbrace{y^H b''}_{=0} - \underbrace{b''^H y}_{=0} + b''^H b'' = y^H y + b''^H b'' \end{aligned}$$

perché gli s.v. sono ortogonali

$$y^H y \geq 0 \rightarrow \text{minimizzo prendendo } y = 0$$

$$\min_{x \in \mathbb{C}^n} \|d(x)\|_2^2 = b''^H b''$$

$$y = 0 \rightarrow Ax^* = b' \quad \text{posso determinare } x^* \text{ perché } b' \in I(A)$$

$$d(x^*) = Ax^* - b' - b'' = -b'' \rightarrow d(x^*) \in I(A)^\perp = N(A^H)$$

$$A^H d(x^*) = 0$$

$$A^H (Ax^* - b) = 0$$

$$A^H A x^* = A^H b$$

$$A^H A x = A^H b$$

SISTEMA DELLE EQUAZIONI NORMALI

(ha sempre soluzione per costruzione)

$$\text{rang}(A^H A | A^H b) = \text{rang}(A^H A)$$

$$N(A) = N(A^H A)$$

$$1) z \in N(A)$$

$$Az = 0 \rightarrow A^H A z = 0 \quad A^H y = 0 \quad \left\{ \begin{array}{l} y \in N(A^H) = I(A)^\perp \\ y = Az \rightarrow y \in I(A) \end{array} \right\}$$

$$2) \left\{ \begin{array}{l} y \in I(A) \rightarrow y = Az \\ y \in I(A)^\perp = N(A^H) \end{array} \right\} \xrightarrow{y=0} Az=0 \rightarrow z \in N(A)$$

$$\left\{ \begin{array}{l} \text{rang}(A) + \dim(N(A)) = n \\ \text{rang}(A^H A) + \dim(N(A^H A)) = n \end{array} \right. \Rightarrow \text{rang}(A) = \text{rang}(A^H A)$$