

QUOZIENTE DI RAYLEIGH

Funzione $R_A(z) : \mathbb{C}^n \rightarrow \mathbb{C}$, definita per ogni $z \neq 0$ e $A \in \mathbb{C}^{n \times n}$ fissato

$$R_A(z) = \frac{z^* A z}{z^* z}$$

Teorema

Sia $A \in \mathbb{C}^{n \times n}$ normale (cioè $AA^* = A^*A$)

$$U^* A U = D = \text{diag}(\lambda_1, \dots, \lambda_n)$$

$$A = U D U^*$$

$$R_A(z) = \frac{z^* A z}{z^* z} = \frac{z^* U D U^* z}{z^* z} = \frac{z^* U D U^* z}{z^* U U^* z}$$

pongo $y = U^* z$
 $y^* = z^* U$

da cui

$$R_A(z) = \frac{y^* D y}{y^* y} = \dots = \frac{\sum_{i=1}^n |y_i|^2 \lambda_i}{y^* y} = \sum_{i=1}^n \frac{|y_i|^2}{y^* y} \lambda_i$$

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pongo $w_i = \frac{|y_i|^2}{y^* y}$ per cui $0 < w_i \leq 1$ e $\sum_{i=1}^n w_i = 1$

da cui

$$R_A(z) = \sum_{i=1}^n w_i \lambda_i$$

considero i valori assoluti:

$$|R_A(z)| = \left| \sum_{i=1}^n w_i \lambda_i \right| \leq \sum_{i=1}^n |w_i| |\lambda_i| \leq \sum_{i=1}^n w_i \rho(A)$$

$$|R_A(z)| \leq \rho(A) \sum_{i=1}^n w_i$$

$$\Rightarrow |R_A(z)| \leq \rho(A)$$