

ESERCIZIO

$$A = \begin{bmatrix} a & -1 & 0 & 0 \\ -1 & a & -1 & 0 \\ 0 & -1 & a & -1 \\ 0 & 0 & -1 & a \end{bmatrix} \quad a \in \mathbb{R}$$

$a = ?$ affinché Jacobi converge

$|a| \geq 2$ A è d.d.d. e $G(A)$ f.c.

$$\Rightarrow \rho(H_J) < 1 \quad \text{e} \quad \rho(H_{GS}) < 1$$

$$H_J = D^{-1}(E+F)$$

$$H_J = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

D^{-1} $E+F$

10

$$\sigma_2 = \det \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} = -\frac{3}{2}$$

$$(\lambda)^4 = \lambda^4 = 1 \quad \sigma_1 = 0 \quad \sigma_3 = 0$$

$$p_H(\lambda) = \lambda^4 - \frac{3}{2}\lambda^2 + \frac{1}{4}$$

$$\lambda^2 = \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{1}{4}}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

$$\lambda_{1,2,3,4} = \pm \sqrt{\frac{3 \pm \sqrt{5}}{2}} \rightarrow \text{tutti reali} \quad (\text{era ovvio perché la matrice è simmetrica, hermitiana})$$

$$|\lambda|_{\max} < 1$$

$$\rho(H_J) = \sqrt{\frac{3+\sqrt{5}}{2}} < 1$$

$$\frac{3+\sqrt{5}}{2} < 1$$

$$\frac{3+\sqrt{5}}{2} < 0^2$$

$$|\lambda| \sqrt{\frac{3+\sqrt{5}}{2}} \simeq 1.618$$

$$-1.618 < \lambda < 1.618$$

$$\rho(H_J) = \sqrt{\frac{3+\sqrt{5}}{2}} \rightarrow \rho(H_{GS}) = \frac{3+\sqrt{5}}{2a^2} \simeq \frac{2.618}{a^2} < 1 \rightarrow a^2 > 2.618$$

$$\|H_J\|_{\infty} = \frac{2}{|a|} < 1 \quad \rightarrow \quad a > 2$$

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$$A = \begin{bmatrix} 1 & a & -a \\ 1 & 1 & 1 \\ a & a & 1 \end{bmatrix} \quad a \in \mathbb{R}$$

$a = ?$ GS e J convergono

$$H_J = D^{-1}(E+F) = \begin{bmatrix} 0 & -a & a \\ -1 & 0 & -1 \\ -a & -a & 0 \end{bmatrix} \rightarrow \text{è degenere}$$

calcoli:

$$\lambda_1 = 0$$

$$\lambda_2 = \sqrt{2a - a^2}$$

$$\lambda_3 = -\lambda_2 = -\sqrt{2a - a^2}$$

intervallo di convergenza del metodo di Jacobi

$$\rho(H_J) = \sqrt{|2a - a^2|} < 1 \rightarrow a \in (1 - \sqrt{2}, 1 + \sqrt{2}), a \neq 1$$

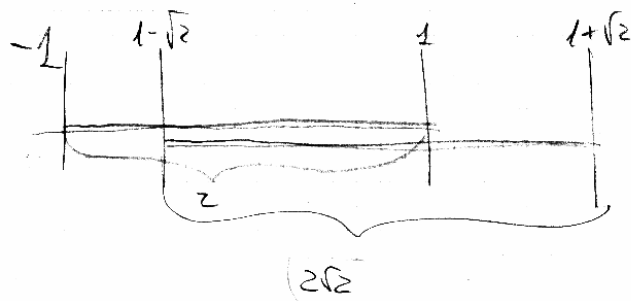
$$H_{GS} = (D-E)^{-1}F =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -a & -a & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -a & a \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -a & 1 \end{bmatrix} \begin{bmatrix} 0 & -a & a \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a & a \\ 0 & a & -a-1 \\ 0 & 0 & a \end{bmatrix}$$

calcoli:

$$\rho(H_{GS}) < 1 \Leftrightarrow |a| < 1 \quad -1 < a < 1$$

Jacobi ha un intervallo di convergenza più ampio



ESERCIZIO

$$\begin{cases} Ax = b \\ x = H(\alpha)x + c \end{cases} \quad \text{due sistemi}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$H(\alpha) = \frac{1}{4} \begin{bmatrix} 1+\alpha & 1-\alpha \\ 1-\alpha & 1+\alpha \end{bmatrix} \quad c = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$\alpha = ?$ affinché i sistemi sono equivalenti

$$\begin{aligned} Ax &= b \\ \text{calcoli: } \downarrow \\ x^* &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

$$x = H(\alpha)x + c$$

$$x^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ è soluz.}$$

$$\frac{1}{4} \begin{bmatrix} 1+\alpha & 1-\alpha \\ 1-\alpha & 1+\alpha \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

non posso ancora dire che i sistemi sono equivalenti, perché non so se x^* è l'unica soluzione.

$$x - H(\alpha)x = c$$

$$(I - H(\alpha))x = c \quad x^* \text{ sarà unica se } \det(I - H(\alpha)) \neq 0$$

$$\Rightarrow \alpha \neq 2 \quad \rightarrow \text{sistemi equivalenti}$$

$$\text{se } \alpha = 2 \quad \rightarrow \text{il secondo sistema ha anche altre soluz.}$$

$$x^{(k+1)} = H(\alpha)x^{(k)} + c \quad \rho(H(\alpha)) < 1$$

calcoli

$$\lambda_1 = \frac{1}{2}$$

$$\lambda_2 = \frac{1}{2}\alpha$$

si converge quando $|\lambda_1|$ e $|\lambda_2|$ sono < 1

$$\Rightarrow \left| \frac{1}{2}\alpha \right| < 1 \rightarrow -2 < \alpha < 2$$

intervallo di convergenza

quando $\rho(H(\alpha))$ è min

$$\rho(H(\alpha)) \text{ min per } \alpha \in (-1, 1) \text{ e } \rho(H(\alpha))_{\text{min}} = \frac{1}{2}$$